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Steady Ship Waves at Low Froude Numbers (Part Two)

by

Francis Noblesse

DTNSRDC-86/078 Steady Ship Waves at Low Froude Numbers (Part Two)

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The report presents a low-Froude-number asymptotic expansion for the far-field wave-amplitude function defined within the Neumann-Kelvin theoretical framework. This asymptotic expansion provides a simple analytical approximation defined explicitly in terms of the geometrical characteristics of the ship hull and the disturbance velocity vector. The low-Froude-number analysis presented in this report shows that the wave resistance and the far-field wave pattern of a ship strongly depend on the shape of the hull, notably the presence of flare and the shape of the waterline at the bow and the stern. In particular, the analysis predicts that the nondimensional wave-resistance coefficient is $O(F^2)$, where F is the Froude number, for a ship form with a region of flare, $O(F^4)$ for a ship form that is wall sided everywhere and has either a bow or a stern (or both) that is neither cusped nor round, and $O(F^6)$ for a wall-sided ship form with both bow and stern that are either cusped or round.

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The analysis also shows that the relative importance of the nonlinear terms in the free-surface boundary condition depends on the shape of the hull. Specifically, the contribution of the nonlinear terms in the free-surface boundary condition to the far-field wave-amplitude function, K, is found to be $O(F^3)$, irrespectively of the hull form. This contribution must be compared with the result predicted by the usual Neumann-Kelvin theory, in which the free-surface boundary condition is linearized. This linear theory predicts that K is $O(F)$ for a ship form having a region of flare, $O(F^2)$ for a ship form that is wall sided everywhere and has either a bow or a stern (or both) that is neither cusped nor round, and $O(F^3)$ for a wall-sided ship form with both bow and stern that are either cusped or round. The contribution of the nonlinear terms in the free-surface boundary condition is then asymptotically negligible for ship forms having a region of flare or a sharp wedge-like bow or/and stern, as is the case for a majority of ships (including naval ships).

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The report presents a low-Froude-number asymptotic expansion for the far-field wave-amplitude function defined within the Neumann-Kelvin theoretical framework. This ~~asymptotic~~ expansion provides a simple analytical approximation defined explicitly in terms of the geometrical characteristics of the ship hull and the disturbance velocity vector. ~~The~~ low-Froude-number analysis ~~presented in this report~~ shows that the wave resistance and ~~the~~ far-field wave pattern of a ship strongly depend on the shape of the hull, notably the presence of flare and the shape of the waterline at the bow and ~~the~~ stern. ~~In particular, the~~ analysis predicts that the nondimensional wave-resistance coefficient is $O(F^2)$*, where F is the Froude number, for a ship form with a region of flare,* $O(F^4)$ *for a ship form that is wall sided everywhere and has either a bow or a stern (or both) that is neither cusped nor round, and* $O(F^6)$ *for a wall-sided ship form with both bow and stern that are either cusped or round.*

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INTRODUCTION

The wave resistance of a ship advancing at constant speed in calm water and the amplitude of the transverse and divergent waves in its far-field wave pattern are defined in terms of the far-field wave-amplitude function, K(t), by means of simple expressions, which may be found in Noblesse [1,2] for instance. The function K(t) is given by the sum of two integrals, namely a surface integral over the ship's mean wetted-hull surface and a line integral along its waterline, involving the disturbance velocity potential in their integrands. An approximate expression for the function K(t), valid for sufficiently small values of the Froude number, was recently obtained in Noblesse [3] by approximating the hull-surface integral by a line integral along the ship waterline and combining the latter waterline integral with that already present in the original unapproximated expression for the function K(t). The resulting low-Froude-number approximate expression in terms of a waterline integral obtained in [3] is now given.

Nondimensional coordinates $(x,y,z) = (X,Y,Z)/L$, where L is the length of the ship, attached to the moving ship are defined. The z axis is taken vertical and pointing upwards and the x axis is chosen in the ship centerplane (port-and-starboard symmetry is assumed here) and pointing towards the bow. The flow disturbance due to the ship is represented in terms of the nondimensional velocity potential $\phi = \overline{\Phi}/UL$, where U is the speed of the ship.

The positive half of the mean waterline is represented by the parametric equations

$$x = x_0(\lambda) \quad \text{and} \quad y = y_0(\lambda)\,, \tag{1a,b}$$

where the parameter λ varies between its bow and stern values, that is

$$\lambda_B \leqslant \lambda \leqslant \lambda_S\,. \tag{1c}$$

In the vicinity of the mean free surface, the positive half of the hull surface is represented by the parametric equations

$$x = x_0(\lambda) + zx_1(\lambda) + z^2x_2(\lambda) + \dots\,, \tag{2a}$$

$$y = y_0(\lambda) + zy_1(\lambda) + z^2y_2(\lambda) + \dots\,, \tag{2b}$$

$$\text{where} \quad \lambda_B \leqslant \lambda \leqslant \lambda_S \text{ and } z \leqslant 0\,. \tag{2c}$$

The velocity potential $\phi(\lambda,z)$ on the hull surface in the vicinity of the plane $z = 0$ likewise is expressed in the form

$$\phi = \phi_0(\lambda) + z\phi_1(\lambda) + z^2\phi_2(\lambda) + \dots\,. \tag{3}$$

Differentiation of the functions $x_n(\lambda)$, $y_n(\lambda)$, $\phi_n(\lambda)$ with respect to the parameter λ is denoted by the superscript $'$; thus, we have $x_0' = dx_0(\lambda)/d\lambda$.

The following low-Froude-number approximation to the far-field wave-amplitude function $K(t)$ is given in [3]:

$$K(t) \sim q^2 (K_+ + K_-) \quad \text{as} \quad F \to 0\,, \tag{4}$$

where the port-and-starboard contributions $K_\pm$ are given by

$$K_\pm = \int_{\lambda_B}^{\lambda_S} \exp\left(-i\theta_\pm/qF^2\right) a_\pm \, d\lambda\,. \tag{4a}$$

In these equations, F is the Froude number defined as

$$F = U/(gL)^{1/2}\,, \tag{5}$$

and we have

$$q = 1/p \quad \text{with} \quad p = (1+t^2)^{1/2} \quad \text{and} \quad 0 \leqslant q \leqslant 1\,; \tag{6a,b,c}$$

furthermore, $\theta_\pm$ and $a_\pm$ are the phase and amplitude functions given by

$$\theta_\pm = x_0 \pm t y_0 \quad \text{and} \tag{7}$$

$$a_\pm = u_\pm a_1^\pm + F^2 q^2 (u_\pm)^2 a_2^\pm + O(F^4)\,, \tag{8}$$

where $u_\pm$ is defined as

$$u_\pm = 1/[1 - iq(x_1 \pm t y_1)]\,, \tag{9}$$

and the functions $a_1^\pm$ and $a_2^\pm$ are now defined.

The first-order amplitude function $a_1^\pm$ is given by

$$\begin{aligned} a_1^\pm = {} & y_0' A_\pm/(1+\varepsilon^2) + 2q(x_0' \pm t y_0')(u_\pm)^2 B_\pm \phi_0 \\ & + C_\pm \phi_0' + u D_\pm \phi_1/(1+\varepsilon^2) + ip(y_1 \phi_0' - y_0' \phi_1)\,, \end{aligned} \tag{10}$$

where we have

$$\varepsilon = (y_0' x_1 - x_0' y_1)/u\,, \tag{11}$$

with u defined as

$$u = [(x_0')^2 + (y_0')^2]^{1/2}\,, \tag{12}$$

and the coefficients $A_\pm$, $B_\pm$, $C_\pm$ and $D_\pm$ are given by

$$A_\pm = [(1+py_0'/u)(1-py_0'/u)+\varepsilon^2] + i(py_0'/u)[y_1(x_0'\pm ty_0')/u+\varepsilon] , \tag{13a}$$

$$B_\pm = q(y_2\mp tx_2) + i(y_1x_2-x_1y_2) , \tag{13b}$$

$$C_\pm = [(\mp t-p^2x_0'y_0'/u^2) + (py_0'/u)^2\varepsilon(x_1x_0'+y_1y_0')/u(1+\varepsilon^2)] \\ + i[y_1(x_0'\pm ty_0')/u+\varepsilon][px_0'/u-(py_0'/u)\varepsilon(x_1x_0'+y_1y_0')/u(1+\varepsilon^2)] , \tag{13c}$$

$$D_\pm = [(x_0'\pm ty_0')/u][(1+\varepsilon^2)(y_1\pm iqt)u_\pm + i(py_0'/u)\varepsilon y_1] - (py_0'/u)\varepsilon(py_0'/u-i\varepsilon). \tag{13d}$$

The second-order amplitude function $a_2^\pm$ is given by

$$a_2^\pm = -y_1' + 2u_\pm(\phi_0m_2^\pm+\phi_1m_1^\pm+\phi_2m_0^\pm-iy_0'\gamma_2^\pm) \\ + 6i(u_\pm)^2[\phi_0(m_1^\pm\gamma_2^\pm+m_0^\pm\gamma_3^\pm)+\phi_1m_0^\pm\gamma_2^\pm] - 12(u_\pm)^3\phi_0m_0^\pm(\gamma_2^\pm)^2 , \tag{14}$$

where we have

$$\gamma_n^\pm = q(x_n\pm ty_n) \quad \text{and} \quad m_n^\pm = \mu_n + iq(y_n'\mp tx_n') , \tag{15a,b}$$

with

$$\mu_0 = x_1y_0'-y_1x_0' = \varepsilon u , \tag{16a}$$

$$\mu_1 = x_1y_1'-y_1x_1' + 2(x_2y_0'-y_2x_0') , \tag{16b}$$

$$\mu_2 = x_1y_2'-y_1x_2' + 2(x_2y_1'-y_2x_1') + 3(x_3y_0'-y_3x_0') . \tag{16c}$$

At a point where the phase of the trigonometric function $\exp(-i\theta_\pm/qF^2)$ is stationary, that is at a point $(x_0 , y_0 , 0)$ where we have $x_0'\pm ty_0' = 0$, the first-order amplitude function $a_1^\pm$ takes the form

$$a_1^\pm = \pm\varepsilon[iy_0'+(py_1\mp it)\phi_0'\mp u\phi_1]/(1\pm i\varepsilon) + ip(y_1\phi_0'-y_0'\phi_1) \quad \text{if } x_0'\pm ty_0' = 0 . \tag{17}$$

Furthermore, if the hull intersects the plane $z = 0$ orthogonally at a point of stationary phase we have

$$a_1^\pm = iF^2py_0'\ \partial^2\phi_0/\partial x^2 \quad \text{if } x_0'\pm ty_0' = 0 \text{ and } n_z = 0 , \tag{18}$$

and the amplitude function $a_\pm$ then is of order F^2.

The low-Froude-number approximation to the far-field wave-amplitude function $K(t)$ given by the waterline integral (4) is well suited for numerical evaluation, as is shown in [3]. However, expression (4) can be simplified by applying the method of stationary phase, which takes advantage of the rapid oscillations of the trigonometric function $\exp(-i\theta_\pm/qF^2)$ in the low-Froude-number limit $F \to 0$. General presentations of the method of stationary phase may be found in Erdélyi [4] and Copson [5]. A brief presentation of this well-known method of approximation is included in the following section for easy reference.

THE METHOD OF STATIONARY PHASE

Let us consider the integral

$$I = \int_{t_1}^{t_2} \exp[i\nu\theta(t)]\, a(t)\, dt\,, \tag{19}$$

where ν is a large positive real variable and the phase-function $\theta(t)$ is real; both the phase-function $\theta(t)$ and the amplitude-function are assumed to be differentiable in the range of integration $[t_1\,,\, t_2]$ to the order required by the analysis. We seek an asymptotic expansion of the integral (19) in the limit $\nu \to \infty$. Due to cancellations between the positive and negative values of the rapidly-oscillating function $\exp[i\nu\theta(t)]$, the major contributions to the value of the integral I stem from the immediate vicinity of the end points t_1 and t_2 of the integration range, on one hand, and from the vicinity of those points where the phase-function $\theta(t)$ is stationary, that is where $\theta'(t) = 0$, on the other hand.

The contribution of the end points may be determined by supposing that the phase-function $\theta(t)$ is not stationary within the integration range, that is we now assume $\theta'(t) \neq 0$ for $t_1 \leqslant t \leqslant t_2$. By integrating the integral (19) by parts, we may obtain

$$I = \frac{1}{i\nu}\frac{a(t)}{\theta'(t)}\, e^{i\nu\theta(t)}\Big|_{t_1}^{t_2} - \frac{1}{i\nu}\int_{t_1}^{t_2} e^{i\nu\theta}\left(\frac{a}{\theta'}\right)' dt\,.$$

Another integration by parts yields the two-term expansion

$$I \sim \frac{1}{i\nu}\frac{a(t)}{\theta'(t)}\, e^{i\nu\theta(t)}\Big|_{t_1}^{t_2} + \frac{1}{\nu^2}\frac{(a/\theta')'}{\theta'}\, e^{i\nu\theta}\Big|_{t_1}^{t_2} + O\!\left(\frac{1}{\nu^3}\right),$$

which may be expressed in the form

$$I \sim \frac{-i}{\nu\theta'}\left[a + \frac{i}{\nu}\left(\frac{a}{\theta'}\right)' + O\!\left(\frac{1}{\nu^2}\right)\right] \exp(i\nu\theta)\Big|_{t_1}^{t_2} \quad \text{as } \nu \to \infty\,. \tag{20}$$

If the functions $\theta(t)$ and $a(t)$ are sufficiently smooth at the end points, higher-order terms in the asymptotic expansion (20) can be obtained by continuing the process of integrating by parts.

Let us now assume that the phase-function $\theta(t)$ is stationary at the interior point $t = T$, where $t_1 < T < t_2$; we then suppose $\theta'(T) = 0$, but $\theta''(T) \neq 0$. The notation

$$\Theta = \theta(T)\,,\ \Theta'' = \theta''(T)\,,\ \ldots \quad A = a(T)\,,\ A' = a'(T)\,,\ \ldots \tag{21}$$

will be used for simplicity. Let the function $\varphi(t)$ be defined as $\varphi(t) = \theta(t) - \theta(T) = \theta(t) - \Theta$. By performing the change of variable $\tau = t - T$ we may then express the integral I in the form

$$I = \exp(i\nu\Theta)\, J\,, \tag{22}$$

where J is the integral defined as

$$J = \int_{-\tau_1}^{\tau_2} \exp[i\nu\varphi(T+\tau)]\, a(T+\tau)\, d\tau \tag{23}$$

with $\tau_1 = T - t_1$ and $\tau_2 = t_2 - T$, so that we have $\tau_1 > 0$ and $\tau_2 > 0$. By expanding the functions $\varphi(T+\tau)$ and $a(T+\tau)$ in Taylor series about the origin $\tau = 0$ we may obtain

$$\varphi(T+\tau) = \tau^2\Theta''/2 + \tau^3\Theta'''/6 + \tau^4\Theta^{(4)}/24 + \ldots$$

$$\text{and} \quad a(T+\tau) = A + \tau A' + \tau^2 A''/2 + \ldots\,. \tag{24}$$

The change of variable $\varphi(T+\tau) = u^2\Theta''/2$ yields

$$u^2 = \tau^2 + \tau^3\Theta'''/3\Theta'' + \tau^4\Theta^{(4)}/12\Theta'' + \ldots\,.$$

By inverting this series we may obtain

$$\tau = u - c_2u^2/3 + c_3u^3/3 + \ldots\,, \tag{25}$$

where the coefficients c_2 and c_3 are given by

$$c_2 = \Theta'''/2\Theta'' \quad \text{and} \quad c_3 = [5(\Theta''')^2/3\Theta'' - \Theta^{(4)}]/8\Theta''\,. \tag{26a,b}$$

By performing the change of variable $\varphi(T+\tau) = u^2\Theta''/2$ in the integral (23) we may obtain

$$J = \int_{-u_1}^{u_2} \exp[i\nu\Theta''u^2/2]\, \alpha(u)\, du\,, \tag{27}$$

where the function $\alpha(u)$ is defined as

$$\alpha(u) = a(T+\tau)\, d\tau/du\,. \tag{28}$$

By using the Taylor series (24) and (25) in equation (28) we may obtain

$$\alpha(u) = A + u(A' - 2c_2A/3) + u^2(A''/2 - c_2A' + c_3A) + \ldots\,. \tag{29}$$

Equations (27) and (29) then yield

$$J = AJ_0 + (A' - 2c_2A/3)J_1 + (A''/2 - c_2A' + c_3A)J_2 + \ldots\,, \tag{30}$$

where J_n are the integrals defined as

$$J_n = \int_{-u_1}^{u_2} \exp[i\nu\Theta''u^2/2]\, u^n\, du\,. \tag{31}$$

The change of variable $\nu\Theta''u^2/2 = \varepsilon t^2$, where ε is defined as

$$\varepsilon = \text{sign}\,\Theta''\,, \tag{32}$$

yields

$$J_n = (2/\nu|\Theta''|)^{(n+1)/2}\, I_n\,, \tag{33}$$

where the integral I_n is defined as

$$I_n = \int_{-t_1}^{t_2} \exp(i\varepsilon t^2)\, t^n\, dt$$

with $t_i = (\nu|\Theta''|/2)^{1/2}u_i$. We then have $t_i \rightarrow \infty$ as $\nu \rightarrow \infty$. The integral I_0 is given by

$$I_0 \sim \int_{-\infty}^{\infty} \exp(i\varepsilon t^2)\, dt = \pi^{1/2} \exp(i\varepsilon\pi/4)\,. \tag{34}$$

The integral I_n is not defined in the limit $t_i \rightarrow \infty$ for $n \geqslant 1$. However, the contribution of the immediate vicinity of the point of stationary phase $t = T$ to the integral (19) can be evaluated by expressing the integral I_n in the form

$$I_n = \int_{-\infty}^{\infty} \exp(i\varepsilon t^2)\, t^n\, \sigma(t)\, dt\,, \tag{35}$$

where the function $\sigma(t)$ is equal to 1 for finite values of t but vanishes exponentially as $t \rightarrow \infty$. By performing an integration by parts we may obtain

$$2I_1 = -i\varepsilon \exp(i\varepsilon t^2)\, \sigma(t) \Big|_{-\infty}^{\infty} + i\varepsilon \int_{-\infty}^{\infty} \exp(i\varepsilon t^2)\, \sigma'(t)\, dt\,. \tag{36}$$

This yields

$$I_1 = 0 \tag{37}$$

since we have $\sigma(t) \rightarrow 0$ as $t \rightarrow \pm\infty$ and $\sigma'(t) \equiv 0$. We may similarly obtain

$$2I_2 = -i\varepsilon \exp(i\varepsilon t^2)\, t\sigma(t) \Big|_{-\infty}^{\infty} + i\varepsilon \int_{-\infty}^{\infty} \exp(i\varepsilon t^2)\, d[t\sigma(t)]\,. \tag{38}$$

This yields

$$2I_2 = i\varepsilon \int_{-\infty}^{\infty} \exp(i\varepsilon t^2)\, \sigma(t)\, dt = i\varepsilon I_0\,, \tag{39}$$

as may be seen from Eq. (34). By using Eqs. (34), (37) and (39) in Eq. (33) we may obtain

$$J_0 = (2\pi/\nu|\Theta''|)^{1/2} \exp(i\varepsilon\pi/4)\,,\ J_1 = 0\,,\ J_2 = iJ_0/\nu\Theta''\,. \tag{40a,b,c}$$

Equations (40a,b,c), (30) and (22) then yield

$$I \sim (2\pi/\nu|\Theta''|)^{1/2} \exp(i\nu\Theta + i\varepsilon\pi/4)\,[A + iA_2/2\nu + O(1/\nu^2)] \quad \text{as} \quad \nu \to \infty\,, \tag{41}$$

where A_2 is defined as

$$A_2 = (A'' - 2c_2A' + 2c_3A)/\Theta''.$$

Equations (26a,b) finally yield

$$A_2 = [(a'/\theta'')' + a\{5(\theta''')^2/3\theta'' - \theta^{(4)}\}/4(\theta'')^2]_{t=T}\,. \tag{42}$$

The contribution of an interior point of stationary phase $t = T$, with $t_1 < T < t_2$, thus is given by Eq. (41), where ε is equal to the sign of $\theta''(T)$ and A_2 is defined by Eq. (42).

An analytical approximation to the integral (19), valid in the limit $\nu \to \infty$, is then given by the sum of the asymptotic expansion (20), corresponding to the contribution of the end points t_1 and t_2, and of the asymptotic expansion (41) for each interior point of stationary phase $t = T$, with $t_1 < T < t_2$. Equations (20) and (41) show that the contributions of the end points and of a point of stationary phase are $O(1/\nu)$ and $O(1/\nu^{1/2})$ as $\nu \to \infty$, respectively. The major contribution to the integral (19) therefore stems from the end points only if there is no point of stationary phase within the integration range. The asymptotic expansions (20) and (41) are not valid in the special case when the end point t_1 or t_2 is a point of stationary phase, that is if we have $\theta'(t_1) = 0$ or $\theta'(t_2) = 0$. In this special case, the two asymptotic expansions should be replaced by the expansion given below.

Let us suppose that the phase-function $\theta(t)$ is stationary at the end point t_1 or t_2, so that we have $\theta'(t_i) = 0$, but $\theta''(t_i) \neq 0$. The contribution of this end point of stationary phase may easily be obtained from the foregoing analysis for an interior point of stationary phase. More precisely, the contribution of the end point t_i is given by equations (22), (30), (33) and (35) where the range of integration $[-\infty, \infty]$ must simply be replaced by $[0, \infty]$ or $[-\infty, 0]$ if the phase is stationary at the lower or upper end point t_1 or t_2, respectively. It may then be verified that the values of the integrals I_0 and I_2 become equal to half the values given by Eqs. (34) and (39), and the value of the integral I_1 is equal to $\mp i\varepsilon/2$ where the upper sign $(-)$ and the lower sign $(+)$ correspond to the cases when t_2 and t_1 are stationary points, respectively. The contribution of the end point t_i when the phase is stationary there is then given by

$$I \sim (\pi/2\nu|\Theta''|)^{1/2} \exp(i\nu\Theta+i\varepsilon\pi/4)$$
$$[A \mp i\varepsilon \exp(-i\varepsilon\pi/4)A_1/(2\nu)^{1/2} + iA_2/2\nu + O(1/\nu^{3/2})] \text{ as } \nu \to \infty , \qquad (43)$$

where ε is equal to the sign of θ'' at the end point, the upper sign $(-)$ and the lower sign $(+)$ correspond to the cases when the phase is stationary at the upper and lower end point, respectively, A_1 is defined as

$$A_1 = 2[(a'-a\theta'''/3\theta'')/(\pi|\theta''|)^{1/2}]_{t=t_i} , \qquad (44)$$

and A_2 is given by Eq. (42) with $T = t_i$.

The asymptotic expansion (41) for the contribution of an interior point of stationary phase $t = T$ is valid if the first derivative $\theta'(t)$ of the phase function $\theta(t)$ vanishes at the point $t = T$ but the second derivative $\theta''(t)$ is nonzero there, as was already noted. However, the asymptotic expansion (41) is clearly not valid if both the first and the second derivatives vanish at $t = T$. An asymptotic expansion for the contribution of an interior point of stationary phase $t = T$, with $t_1 < T < t_2$, in the case when $\theta'(T) = 0$ and $\theta''(T) = 0$, but $\theta'''(T) \neq 0$, can be obtained in a manner similar to that used for obtaining the asymptotic expansion (41). The change of variable $\varphi(T+\tau) = u^2\Theta''/2$ used below Eq. (24) and prior to Eq. (27) is replaced by $\varphi(T+\tau) = u^3\Theta'''/6$, and Eqs. (25) and (26a,b) become

$$\tau = u - c_2u^2/6 + c_3u^3/6 + \dots \text{ with} \qquad (45)$$

$$c_2 = \Theta^{(4)}/2\Theta''' \text{ and } c_3 = [(\Theta^{(4)})^2/4\Theta'''-\Theta^{(5)}/5]/2\Theta''' . \qquad (46a,b)$$

Equations (30) and (31) then take the form

$$J = AJ_0 + (A'-c_2A/3)J_1 + (A''-c_2A'+c_3A)J_2/2 + \dots \qquad (47)$$

$$\text{with } J_n = \int_{-u_1}^{u_2} \exp[i\nu\Theta'''u^3/6]\, u^n\, du . \qquad (48)$$

The change of variable $\nu\Theta'''u^3/6 = t^3$ yields

$$J_n = (6/\nu\Theta''')^{(n+1)/3}\, I_n \qquad (49)$$

$$\text{with } I_n = \int_{-\infty}^{\infty} \exp(it^3)\, t^n\sigma(t)\, dt . \qquad (50)$$

It may be verified that we have

$$I_0 = \Gamma(1/3)/3^{1/2} , \ I_1 = i2\pi/3\Gamma(1/3) \text{ and } I_2 = 0 , \qquad (51a,b,c)$$

where $\Gamma(1/3) = 2.6789\dots$. Equations (22), (47), (49), (51a,b,c) and (46a,b) finally yield the asymptotic expansion

$$I \sim 3^{-1/2}\Gamma(1/3)\,(6/\nu\Theta''')^{1/3}\exp(i\nu\Theta)\,[A + iA_2/\nu^{1/3} + O(1/\nu)] \quad \text{as } \nu \to \infty \tag{52}$$

where A_2 is defined as

$$A_2 = [\pi 2^{4/3} / 3^{1/6}\{\Gamma(1/3)\}^2]\,[(a' - a\theta^{(4)}/6\theta''')/(\theta''')^{1/3}]_{t=T}\,. \tag{53}$$

This asymptotic expansion defines the contribution of an interior point of stationary phase $t = T$, with $t_1 < T < t_2$, where we have $\theta'(T) = 0 = \theta''(T)$ but $\theta'''(T) \neq 0$.

BOW-AND-STERN CONTRIBUTION

Equations (4a), (19) and (20) show that the contribution of the bow and the stern to the low-Froude-number asymptotic approximation to the integral $K_\pm$ is given by

$$K_\pm \sim (iF^2q/\theta'_\pm)\,[a_\pm - iF^2q(a_\pm/\theta'_\pm)' + O(F^4)]\exp(-ip\theta_\pm/F^2)\Big|_{\lambda_B}^{\lambda_S}\,.$$

By using Eqs. (4), (7) and (8) we may then obtain

$$K \sim iF^2q^3\,[A_B\exp(-ipx_B/F^2) - A_S\exp(-ipx_S/F^2)]\,, \tag{54}$$

where $x_{B,S}$ are the abscissae of the bow and the stern and the bow-and-stern amplitude functions $A_{B,S}$ are defined as

$$A_{B,S} = 2n_xA_1 + iF^2q(A_2^+ + A_2^-) + O(F^4)\,, \tag{55}$$

where the expression on the right side is evaluated at the bow or at the stern. In this equation, the second-order amplitude functions $A_2^\pm$ are given by

$$A_2^\pm\,\theta'_\pm = (u_\pm a_1{}^\pm/\theta'_\pm)' + iq(u_\pm)^2a_2{}^\pm\,, \tag{56}$$

and the first-order amplitude function A_1 is defined as

$$-2n_x\,A_1 = u_+a_1^+/\theta'_+ + u_-a_1^-/\theta'_- \tag{57}$$

for reasons that will become clear further on.

Equation (2b) yields $y_1 = 0$ since we have $y \equiv 0$ at the bow or the stern. Equation (9) thus becomes

$$u_+ = 1/(1 - iqx_1) = u_-\,. \tag{58}$$

Equation (57) then yields

$$-2n_x(1 - iqx_1)\theta'_+\theta'_-A_1 = a_1^+\theta'_- + a_1^-\theta'_+\,.$$

By using Eqs. (7), (10), (58) and the relation $y_1 = 0$ into the foregoing equation, we may obtain

$$-2n_x(1-iqx_1)\theta'_+\theta'_-A_1 = y_0'(A_+\theta'_- + A_-\theta'_+)/(1+\varepsilon^2)$$
$$+\ 2q\,\theta'_+\theta'_-(B_+ + B_-)\phi_0/(1-iqx_1)^2 + (C_+\theta'_- + C_-\theta'_+)\phi_0'$$
$$+\ u(D_+\theta'_- + D_-\theta'_+)\phi_1/(1+\varepsilon^2) - 2ipx_0'y_0'\phi_1 \,. \tag{59}$$

By using the relation $y_1 = 0$ into Eqs. (13a) and (11) we may obtain

$$A_\pm = (1+py_0'/u)(1-py_0'/u)+\varepsilon^2+i\varepsilon py_0'/u = A \,, \tag{60}$$

with $\quad \varepsilon = x_1y_0'/u \,.$ (61)

We then have

$$A_+\theta'_- + A_-\theta'_+ = A(\theta'_+ + \theta'_-) = 2x_0'A \,,$$

where Eq. (7) was used. By using Eqs. (60) and (61) we may finally obtain

$$A_+\theta'_- + A_-\theta'_+ = 2x_0'[1+(x_1{}^2-p^2+ipx_1)(y_0'/u)^2] \,. \tag{62}$$

By using the relations $y_1 = 0$ and $y_2 = 0$, which follow from Eq. (2b) and the condition $y = 0$ at the bow or the stern, in Eq. (13b) we may obtain $B_\pm = \mp qtx_2$. We then have

$$B_+ + B_- = 0 \,. \tag{63}$$

By using the relation $y_1 = 0$ into Eq. (13c) we may obtain

$$C_\pm = \mp t - (py_0'/u - i\varepsilon)[px_0'/u-(py_0'/u)\varepsilon x_1x_0'/u(1+\varepsilon^2)] \,,$$

which becomes

$$C_\pm = \mp t - (1-iqx_1)(p^2x_0'y_0'/u^2)/(1+\varepsilon^2)$$

by virtue of Eq. (61). By using Eq. (7) we may finally obtain

$$C_+\theta'_- + C_-\theta'_+ = 2y_0'[t^2-(1-iqx_1)(px_0'/u)^2/(1+\varepsilon^2)] \,. \tag{64}$$

Equations (13d), (58), (7) and the relation $y_1 = 0$ yield

$$D_\pm = \pm iqt(1+\varepsilon^2)\theta'_\pm/u(1-iqx_1) - (py_0'/u)\varepsilon(py_0'/u - i\varepsilon) \,.$$

We then have

$$D_+\theta'_- + D_-\theta'_+ = -2p^2 x_1 x_0'(y_0'/u)^3(1-iqx_1)\,, \tag{65}$$

where Eqs. (7) and (61) were used.

By substituting Eqs. (62), (63), (64) and (65) into Eq. (59) we may obtain

$$(1-iqx_1)(1+\varepsilon^2)^{1/2}[(x_0'/u)^2-(ty_0'/u)^2]A_1 = -[1+(x_1{}^2-p^2+ipx_1)(y_0'/u)^2]x_0'/u$$
$$- [t^2+(tx_1y_0'/u)^2-(1-iqx_1)(px_0'/u)^2]\phi_0'/u + ip[1-ipx_1(y_0'/u)^2]\phi_1 x_0'/u\,, \tag{66}$$

where Eqs. (7) and (61) and the relation $n_x = y_0'/u(1+\varepsilon^2)^{1/2}$, given by Eq. (33a) in [3], were used. Equations (31) and the equation below it in [3] yield

$$\phi_1 = x_1 t_x \phi_t - (t_x n_y - t_y n_x + x_1 t_y n_z)\phi_d\,, \tag{67}$$

where the relation $y_1 = 0$ was used. Furthermore, we have

$$\varepsilon = -n_z/(1-n_z{}^2)^{1/2} \quad \text{and} \quad 1+\varepsilon^2 = 1/(1-n_z{}^2)\,, \tag{68a,b}$$

as may be obtained from Eq. (33c) in [3]. Equations (39), (38a,b) and (33a,b) in [3] also yield

$$-\phi_0'/u = \phi_t\,, \tag{69a}$$
$$-x_0'/u = t_x = n_y/(1-n_z{}^2)^{1/2}\,, \tag{69b}$$
$$-y_0'/u = t_y = -n_x/(1-n_z{}^2)^{1/2}\,, \tag{69c}$$

where Eq. (68b) was used. Equations (61) and (69c) give

$$x_1 = -n_z/n_x\,. \tag{70}$$

Equation (67) becomes

$$\phi_1 = -t_x n_z \phi_t/n_x - \phi_d/(1-n_z{}^2)^{1/2} \tag{71}$$

upon using Eqs. (70) and (69b,c). By substituting Eqs. (69a,b,c), (70) and (71) into Eq. (66) we may obtain

$$(n_x+iqn_z)(t_x{}^2-t^2t_y{}^2)A_1 = n_y[n_x{}^2+t_y{}^2(n_z{}^2-p^2n_x{}^2-ipn_xn_z)]/n_x$$
$$-t_y(t^2-p^2t_x{}^2)\phi_t + ipt_x(n_x+ipt_y{}^2n_z)\phi_d\,.$$

By using Eqs. (6b), (69b,c) and the relations $t_x{}^2+t_y{}^2 = 1$ and $n_x{}^2+n_y{}^2+n_z{}^2 = 1$ in the foregoing expression we may obtain

$$(n_x+iqn_z)A_1 = n_xn_y[1-ipn_z(n_x+iqn_z)/(1-n_z^2-p^2n_x^2)]$$
$$+ t_y\phi_t - ipn_yt_y\phi_d[1+p^2n_x(n_x+iqn_z)/(1-n_z^2-p^2n_x^2)] .$$

Upon dividing by n_x+iqn_z we may finally obtain

$$A_1 = n_x(n_y-\psi_t+ipn_y\psi_d)/(n_x+iqn_z)$$
$$+ ipn_xn_y(n_x\psi_d-q^2n_z)/[q^2(1-n_z^2)-n_x^2] , \tag{72}$$

where we have

$$(\psi_t , \psi_d) = (\phi_t , \phi_d)/(1-n_z^2)^{1/2} . \tag{73a,b}$$

The contribution of the bow and the stern to the far-field wave-amplitude function K(t) is then given by Eqs. (54) and (55), where the first- and second-order amplitude functions A_1 and $A_2^{\pm}$ are defined by Eqs. (72) and (56), respectively. Expression (56) for the second-order amplitude function is complex. However, Eq. (72) defines the first-order amplitude function A_1 explicitly in terms of t, $p = (1+t^2)^{1/2}$ and $q = 1/p$, the geometrical characteristics of the hull at the bow and the stern — specifically, the unit outward hull-normal vector $\vec{n}(n_x,n_y,n_z)$ — and the components ϕ_t and ϕ_d of the disturbance velocity vector in the directions of the unit hull-tangent vectors $\vec{t}$ and $\vec{n}\times\vec{t}$.

Equation (55) shows that the first-order approximation to the bow-or-stern amplitude function $A_{B,S}$ — given by $2n_xA_1$ — vanishes if $n_x = 0$, that is if the waterline has a cusp at the bow or the stern. Equation (72) shows that the first-order approximation to the amplitude function $A_{B,S}$ vanishes also if the bow or the stern is round, since we then have $n_y = 0$ and $\phi_t = -\phi_y = 0$ by symmetry. It may thus be seen that the contribution of the bow (or the stern) to the far-field wave-amplitude function K(t) is of order F^2 if $n_x = 0$ or $n_y = 0$ at the bow (or the stern); that is, we have

$$K_{B,S} = O(F^2) \quad \text{if} \quad n_x^{B,S} = 0 \quad \text{or} \quad n_y^{B,S} = 0 . \tag{74}$$

In the particular case when the hull surface is vertical at the bow or the stern, we have $n_z = 0$ and Eq. (72) becomes

$$A_1 = t_x - \phi_t - iqt_x\phi_z/(q^2-t_y^2) \quad \text{if} \quad n_z = 0 , \tag{75}$$

where the relations $\phi_d = -\phi_z$ and $n_y = t_x$ were used. This yields

$$A_1 = t_x - \phi_t + O(F^2) \quad \text{if} \quad n_z = 0 , \tag{76}$$

as is indicated by the free-surface boundary condition $\phi_z = -F^2\phi_{xx} = O(F^2)$.

In the limit $t \to \infty$, we have $q \to 0$ and Eq. (72) yields

$$[\text{illegible}] \rightarrow n_y - (\phi_t - \phi_d n_z n_y/n_x)/(1-n_z^2)^{1/2} \quad \text{as} \quad t \rightarrow \infty . \tag{77}$$

In the particular case $t = 0$, we have $p = 1 = q$ and Eq. (72) becomes

$$A_1 = n_x(n_y - \psi_t + i n_y \psi_d)/(n_x + iqn_z) + in_x(n_x\psi_d - n_z)/n_y \quad \text{for} \quad t = 0 . \tag{78}$$

Expression (78) is not valid if $n_y = 0$, that is if the bow or the stern is round. More generally, expression (72) is not valid if $q^2 = n_x^2/(1-n_z^2) = t_y^2$, that is if $t_x^2 - t^2 t_y^2 = 0$. This special case, in which the bow or the stern is a point of stationary phase, is considered further on.

STATIONARY-PHASE CONTRIBUTION

Equations (4), (4a), (7), (8), (19), (41) and (42) show that the contribution of the point(s) of stationary phase may be expressed in the form

$$K \sim i\, F\, q^3 \sum_{\pm} K_{\pm} \; ; \tag{79}$$

in this expression, $K_{\pm}$ corresponds to the contribution of a point of the mean waterline where the phase of the trigonometric function $\exp[-ip(x_0 \pm ty_0)/F^2]$ is stationary, that is where we have

$$t = \mp dx_0/dy_0 = \mp x_0'/y_0' = \mp t_x/t_y , \tag{80}$$

as may be obtained by using Eqs. (69b,c), and the notation $\sum_{\pm}$ implies summation over all the points of stationary phase.

The stationary-phase contribution $K_{\pm}$ in Eq. (79) is given by

$$K_{\pm} = \pm(2\pi r)^{1/2} A_{\pm} \exp[-i(p\theta_{\pm}/F^2 \mp \varepsilon\pi/4)] , \tag{81}$$

where r and ε are defined as

$$r = u^2/q|x_0'' \pm t y_0''| , \tag{82}$$

$$\varepsilon = \mp \operatorname{sign}(x_0'' \pm t y_0'') ; \tag{83}$$

furthermore, the amplitude-function $A_{\pm}$ is given by

$$A_{\pm} = A_1^{\pm} - F^2 q A_2^{\pm} + O(F^4) , \tag{84}$$

where the first- and second-order amplitude functions $A_1^{\pm}$ and $A_2^{\pm}$ are defined as

$$A_1^{\pm} = \mp i u_{\pm} a_1^{\pm}/u , \tag{85}$$

$$\pm 2uA_2^{\pm} = [(u_{\pm}a_1^{\pm})'/\theta_{\pm}'']' + u_{\pm}a_1^{\pm}[5(\theta_{\pm}''')^2/3\theta_{\pm}'' - \theta_{\pm}^{(4)}]/4(\theta_{\pm}'')^2$$
$$+ 2iq(u_{\pm})^2 a_2^{\pm} . \tag{86}$$

In Eqs. (82), (85) and (86), we have $u = [(x_0')^2 + (y_0')^2]^{1/2}$, as is given by Eq. (12); furthermore, we have $\theta_{\pm} = x_0 \pm ty_0$, as is indicated by Eq. (7), in Eqs. (81) and (86).

Equations (6a,b), (12), (69c) and the stationary-phase relation (80) yield

$$q = \pm y_0'/u = \mp t_y = 1/p , \tag{87}$$

where the condition $x_0' \leqslant 0$, which follows from Eq. (1a), was used. By using Eqs. (7), (80) and (87) we may then obtain

$$p\theta_{\pm} = \pm(y_0 t_x - x_0 t_y)/t_y^2 \quad \text{and} \tag{88}$$

$$r = [(x_0')^2 + (y_0')^2]^{3/2} / |x_0'' y_0' - y_0'' x_0'| , \tag{89}$$

which shows that r is equal to the radius of curvature of the mean waterline at the stationary-phase point (x_0 , y_0).

Let (α , β) be the (x , y) coordinates of the center of curvature of the mean waterline at the point of stationary phase (x_0 , y_0). We have

$$\alpha = x_0 + y_0'[(x_0')^2 + (y_0')^2]/(x_0'' y_0' - y_0'' x_0') .$$

Equations (83) and (80) yield

$$\varepsilon = \mp \operatorname{sign} (x_0'' y_0' - y_0'' x_0')/y_0' .$$

We then have

$$\varepsilon = \mp \operatorname{sign} (\alpha - x_0) , \tag{90}$$

which shows that ε takes the values -1 or $+1$ if the center of curvature corresponding to the stationary-phase point (x_0 , y_0) is upstream or downstream from (x_0 , y_0), respectively. Equation (80) shows that we have $\mp t_y \geqslant 0$ at a point of stationary phase, since we have $t \geqslant 0$ and $t_x \geqslant 0$. More precisely, Eq. (7) shows that the points where the phases θ_+ and θ_- are stationary are located in the fore and aft portions of the mean waterline, where we have $t_y \leqslant 0$ and $t_y \geqslant 0$, respectively. If the mean waterline is convex, we have $\alpha \leqslant x_0$ and $\alpha \geqslant x_0$ in its fore and aft portions, respectively. It may then be seen from Eq. (90) that ε is equal to $+1$ or -1 if the mean waterline is convex or concave, respectively, at the point of stationary phase.

The relation $-x_0'/u = [1 - (y_0'/u)^2]^{1/2}$, which follows from Eq. (12), and Eqs. (87) and (6a,b) yield

$$qt = -x_0'/u = t_x , \tag{91}$$

where Eq. (69b) was used. By using Eqs. (87) and (91) in Eq. (11) we may obtain

$$\varepsilon = \pm q(x_1 \pm t y_1) . \tag{92}$$

Equation (9) then becomes

$$u_\pm = 1/(1 \mp i\varepsilon) . \tag{93}$$

Upon using Eqs. (17), (93), (69a,b,c) and (87), Eq. (85) becomes

$$(1+\varepsilon^2)A_1^\pm = \pm q\varepsilon \pm (t\varepsilon - p y_1)\phi_t - \phi_1 . \tag{94}$$

Equation (31) and the equation below it in [3] yield

$$\phi_1 = (x_1 t_x + y_1 t_y)\phi_t - D\phi_d , \tag{95}$$

where D is defined as

$$D = t_x n_y - t_y n_x + (x_1 t_y - y_1 t_x) n_z . \tag{96}$$

Equations (69b,c), the relation $t_x^2 + t_y^2 = 1$ and Eqs. (11) and (68a,b) show that we have

$$D = (1+\varepsilon^2)^{1/2} . \tag{97}$$

Upon using Eqs. (97), (87) and (91) in Eq. (95) we may obtain

$$\phi_1 = q(t x_1 \mp y_1)\phi_t - (1+\varepsilon^2)^{1/2}\phi_d . \tag{98}$$

It follows from Eqs. (2a,b) that the vector $\partial\vec{x}/\partial z = (x_1 , y_1 , 1)$ is tangent to the hull surface. We then have $\vec{n}\cdot\partial\vec{x}/\partial z = 0$, that is

$$x_1 n_x + y_1 n_y + n_z = 0 . \tag{99}$$

This equation yields

$$x_1 = \mp t y_1 - n_z/n_x , \tag{100}$$

where Eqs. (69b,c) and (80) were used. By using Eqs. (100) and (6a,b) in Eq. (98) we may obtain

$$-\phi_1 = (q t n_z/n_x \pm p y_1)\phi_t + (1+\varepsilon^2)^{1/2}\phi_d . \tag{101}$$

Equations (94) and (101) then yield

$$(1+\varepsilon^2)A_1^{\pm} = \pm q\varepsilon + t(qn_z/n_x \pm \varepsilon)\phi_t + (1+\varepsilon^2)^{1/2}\phi_d \,. \tag{102}$$

Equations (68a), (69c) and (87) show that we have

$$qn_z/n_x \pm \varepsilon = 0 \,.$$

Equation (102) finally becomes

$$A_1^{\pm} = -n_z n_x + (1-n_z^2)^{1/2}\phi_d \,, \tag{103}$$

where Eqs. (68a,b), (69c) and (87) were used.

Upon using Eq. (88) we may express Eq. (81) in the form

$$K_{\pm} = \pm(2\pi r)^{1/2} A_{\pm} \exp[\mp i\{(y_0 t_x - x_0 t_y)/t_y^2 F^2 - \varepsilon\pi/4\}] \,; \tag{104}$$

in this expression for the contribution of a point of stationary phase (x_0 , y_0), which is defined by Eq. (80), r represents the radius of curvature of the mean waterline at the point (x_0 , y_0), ε is equal to $+1$ or -1 if the mean waterline is convex or concave, respectively, at (x_0 , y_0) and $A_{\pm}$ is the amplitude function defined by Eqs. (84), (86) and (103).

Expression (86) for the second-order amplitude function $A_2^{\pm}$ is a complex one. However, Eq. (103) defines the first-order amplitude function $A_1^{\pm}$ explicitly in terms of the geometrical characteristics of the hull — specifically, the components n_x and n_z of the unit outward normal vector $\vec{n}$ to the hull — and the velocity component ϕ_d in the direction of the downward tangential unit vector $\vec{n}\times\vec{t}$ at the point of stationary phase.

Equations (84) and (103) show that in the particular case when the hull surface is vertical at the point of stationary phase, we have

$$A_{\pm} = -\phi_z + O(F^2) \quad \text{if} \quad n_z = 0 \,. \tag{105}$$

The free-surface condition $\phi_z = -F^2\phi_{xx} = O(F^2)$ then yields

$$A_{\pm} = O(F^2) \quad \text{if} \quad n_z = 0 \,. \tag{106}$$

The summation in expression (79) for the contribution of the point(s) of stationary phase is extended to any point of the mean waterline where the phases of the trigonometric functions $\exp[-ip(x_0 \pm t y_0)/F^2]$ are stationary, that is where Eq. (80) holds. The number of stationary points, and their positions along the waterline, depend on the value of t and on the shape of the waterline. For instance, for the simple case of a hull with waterline consisting of a sharp-ended parabolic bow region $1/4 \leqslant x \leqslant 1/2$ defined by the equation $y = 4bx(1-2x)$, where b denotes the ship's beam/length ratio, a straight parallel midbody region $-1/4 \leqslant x \leqslant 1/4$, and a round-ended elliptic stern region $-1/2 \leqslant x \leqslant -1/4$ defined by the equation $y = b[-2x(1+2x)]^{1/2}$, there is one point of stationary phase in the stern region given by $x = -[1+1/(1+4b^2t^2)^{1/2}]/4$, so that we have $-1/2 \leqslant x \leqslant -1/4$ for $0 \leqslant t \leqslant \infty$ with $x \to -1/2$ as $t \to 0$ and $x \to -1/4$ as $t \to \infty$, and one point of stationary phase in the bow region given by $x = (1+1/4bt)/4$ for $1/4b \leqslant t \leqslant \infty$, so that we have $1/2 \geqslant x \geqslant 1/4$ for $1/4b \leqslant t \leqslant \infty$ with $x \to 1/2$ as $t \to 1/4b$ and $x \to 1/4$ as $t \to \infty$. We thus have one point of stationary phase in the stern region for $0 \leqslant t < 1/4b$ and two points of stationary phase, one in the stern region and one in the bow region, for $1/4b \leqslant t \leqslant \infty$. The two points of stationary phase approach the shoulders $x = \pm 1/4$, where $dy/dx = 0$, in the limit $t \to \infty$.

Expression (104) is not valid at a point of stationary phase of order two, for which both the first and the second derivatives of the phase vanish. In this case, we thus have

$$x_0' \pm t y_0' = 0 \quad \text{and} \quad x_0'' \pm t y_0'' = 0 \; .$$

These two conditions can be satisfied simultaneously at a point $(x_0 \, , \, y_0)$ where we have

$$x_0'' y_0' - y_0'' x_0' = 0 \; ,$$

that is at a point of inflexion of the mean waterline, at which the radius of curvature is infinite as may be seen from Eqs. (89) or (82). This particular case is considered further on.

SUMMARY OF RESULTS

Equations (79) and (54) yield the following low-Froude-number asymptotic expansion for the far-field wave-amplitude function K(t):

$$K(t) \sim iFq^3 \, [\, \sum_{\pm} K_{\pm} + F(K_B - K_S) \,] \quad \text{as} \quad F \to 0 \; , \tag{107}$$

where F is the Froude number defined by Eq. (5), that is:

$$F = U/(gL)^{1/2} \; , \tag{108}$$

q is given by Eqs. (6a,b,c), i.e.:

$$q = 1/p \quad \text{with} \quad p = (1+t^2)^{1/2} \quad \text{and} \quad 0 \leqslant q \leqslant 1 \; , \qquad \text{(109a,b,c)}$$

and $K_{\pm}$, K_B and K_S represent the contributions of the point(s) of stationary phase on the mean waterline, and of the bow and the stern, respectively.

The notation $\sum_{\mp}$ implies summation over all the points of the mean waterline where the phases of the trigonometric functions $\exp[-ip(x \pm ty)/F^2]$ are stationary, and the terms $K_{\pm}$ correspond to the contributions of the stationary points of the phases

$x \pm ty$, respectively. These points of stationary phase are defined by the equivalent relations (80), that is:

$$dy/dx = t_y/t_x = \mp 1/t \,, \tag{110}$$

where t_x and t_y are the components of the unit vector $\vec{t}$ tangent to the mean waterline and pointing towards the bow. The number of points of stationary phase, and their positions on the mean waterline, depend on the value of t and on the shape of the waterline. We usually have one or two points of stationary phase for typical hull forms, as was shown in the previous section.

The stationary-phase contribution $K_\pm$ is given by Eq. (104), that is:

$$K_\pm = \pm(2\pi r)^{1/2} A_\pm \exp[\mp i\{(yt_x - xt_y)/t_y{}^2F^2 - \varepsilon\pi/4\}] \;; \tag{111}$$

in this expression, r represents the radius of curvature of the mean waterline at the point of stationary phase $(x\,,\,y)$, ε is equal to $+1$ or -1 if the mean waterline is convex or concave, respectively, at $(x\,,\,y)$ and $A_\pm$ is the amplitude function defined by Eq. (84), that is:

$$A_\pm = A_1 - F^2 q A_2{}^\pm + O(F^4) \,. \tag{112}$$

The first-order amplitude function A_1 is given by Eq. (103), i.e.:

$$A_1 = -n_z n_x + (1-n_z{}^2)^{1/2}\phi_d \,, \tag{113}$$

where $\vec{n}(n_x\,,\,n_y\,,\,n_z)$ is the unit outward normal vector to the hull and ϕ_d is the velocity component in the direction of the downward tangential vector $\vec{n}\times\vec{t}$ to the hull. The second-order amplitude function $A_2{}^\pm$ in Eq. (112) is given by a complex expression, namely Eq. (86), where $\theta_\pm$, $u_\pm$, u , $a_1{}^\pm$ and $a_2{}^\pm$ are defined by Eqs. (7), (9), (12), (17) and (14). Equations (112) and (113) show that in the particular case when the hull surface is vertical at the point of stationary phase, we have

$$A_\pm = -\phi_z + O(F^2) = O(F^2) \quad \text{if} \quad n_z = 0 \,. \tag{114}$$

The bow and stern contributions $K_{B,S}$ in the asymptotic approximation (107) are defined by Eq. (54), that is:

$$K_{B,S} = A_{B,S} \exp(-ip\, x_{B,S}/F^2) \,, \tag{115}$$

where $x_{B,S}$ are the abscissae of the bow and the stern, respectively, and the amplitude functions $A_{B,S}$ are defined by Eq. (55), i.e.:

$$A_{B,S} = 2n_x A_1 + iF^2 q(A_2^+ + A_2^-) + O(F^4) \,. \tag{116}$$

The first-order amplitude function A_1 is given by Eq. (72), that is

$$A_1 = n_x(n_y - \psi_t + ipn_y\psi_d)/(n_x + iqn_z) + ipn_xn_y(n_x\psi_d - q^2n_z)/[q^2(1-n_z^2) - n_x^2] \, , \quad (117)$$

where $(\psi_t \, , \, \psi_d) = (\phi_t \, , \, \phi_d)/(1-n_z^2)^{1/2}$ (118a,b)

and ϕ_t is the velocity component in the direction of the tangent vector $\vec{t}$ to the mean waterline. The second-order amplitude functions $A_2^{\pm}$ in Eq. (116) are given by a complex expression, namely Eq. (56), where $\theta_{\pm}$, $u_{\pm}$, $a_1^{\pm}$ and $a_2^{\pm}$ are defined by Eqs. (7), (9), (10) and (14). Equations (116) and (117) show that we have $n_xA_1 = 0$ and therefore

$$A_{B,S} = O(F^2) \quad \text{if} \quad n_x^{B,S} = 0 \quad \text{or} \quad n_y^{B,S} = 0 \, , \quad (119)$$

that is, if the bow or the stern is either cusped or round.

The asymptotic approximation (107) and Eqs. (111)-(114) and (115)-(119) defining the contributions of the point(s) of stationary phase on the waterline and of the bow and the stern, respectively, show that the low-Froude-number behavior of the far-field wave-amplitude function is strongly influenced by the shape of the hull in the vicinity of the mean free surface. More precisely, for a value of t for which there is one (or more) point of stationary phase on the mean waterline where the hull has flare, the contribution of this stationary-phase point dominates the contributions of the bow and the stern and we have $K(t) = O(F)$. On the other hand, for a value of t for which either there corresponds no point of stationary phase or the hull has no flare (i.e. is vertical) at the point(s) of stationary phase, the contributions of the bow and the stern are dominant, and we have $K(t) = O(F^2)$, except if both the bow and the stern are either cusped or round. In the latter case, the contribution of the point(s) of stationary phase, where the hull is assumed to be vertical, is dominant and we have $K(t) = O(F^3)$.

The wave resistance of a ship is defined in terms of the far-field wave-amplitude function by means of the well-known Havelock integral

$$\pi R/\varrho U^2L^2 = \int_0^{\infty} ||K(t)||^2 p \; dt \, . \quad (120)$$

The low-Froude-number asymptotic approximation (107) then shows that the nondimensional wave-resistance coefficient is $O(F^2)$ for a ship form with a region of flare, $O(F^4)$ for a ship form that is vertical (wall sided) everywhere along its mean waterline and has either a bow or a stern (or both) that is neither cusped nor round, and $O(F^6)$ for an everywhere wall-sided ship form with both bow and stern that are either cusped or round.

In the case of a ship form that has flare over a portion of its waterline and is wall sided elsewhere, the far-field wave-amplitude function K(t) is O(F) for the range of values of t for which there corresponds a point of stationary phase within the region of flare but K(t) is $O(F^2)$ for other values of t, for which the corresponding points of stationary phase fall outside the region of flare. If the portion of the waterline where the ship has flare is of very small extent, the corresponding range of values of t for which the function K(t) is O(F) is also small, so that the function K(t) may exhibit a sharp peak in the limit $F \to 0$. The amplitudes of the transverse and divergent waves in the far-field Kelvin wake of a ship are directly related to the far-field wave-amplitude function K(t) and therefore may also exhibit a sharp peak at an angle from the track of the ship smaller than the Kelvin cusp angle of 19° 1/2, as is shown in Noblesse [2].

SPECIAL CASES

As was already noted, the asymptotic expansion (107) is not uniformly valid for all values of t. In particular, this expansion is not valid in the vicinity of the values of t for which the bow or the stern is a point of stationary phase, that is for $t = t_B$ or t_S, with

$$t_B = -(dx/dy)_{bow} = -(t_x/t_y)_{bow} \,, \tag{121a}$$

$$t_S = (dx/dy)_{stern} = (t_x/t_y)_{stern} \,. \tag{121b}$$

Equations (41) and (43) show that the stationary-phase and bow contributions $K_+ + FK_B$ in Eq. (107) become $K_+/2 + O(F)$ for the special value $t = t_B$. The stationary-phase and stern contributions $K_- - FK_S$ in Eq. (107) likewise become $K_-/2 + O(F)$ for $t = t_S$. We thus have

$$K_+ + FK_B \to K_+/2 + O(F) \quad \text{if} \quad t = t_B \,, \tag{122a}$$

$$K_- - FK_S \to K_-/2 + O(F) \quad \text{if} \quad t = t_S \,, \tag{122b}$$

where $K_\pm$ is given by Eqs. (111)-(113).

Equations (122a,b) and (111)-(113) yield

$$K_\pm \pm FK_{B,S} \to \pm(\pi r/2)^{1/2} A \exp[\pm i(x/t_y F^2 + \pi/4)] + O(F) \quad \text{if} \quad t = t_{B,S} \,, \tag{123}$$

where y and ε in Eq. (111) were taken equal to 0 and 1, respectively (since the waterline must be convex at the bow and at the stern), and the amplitude function A is given by

$$A = -n_z n_x + (1-n_z^2)^{1/2} \phi_d \,. \tag{124}$$

As was noted previously, the asymptotic expansion (107) is also not valid in the vicinity of a point of stationary phase for which the mean waterline has an inflexion point; this corresponds to the case when both the first derivative and the second derivative of the phase vanish, that is we have both $x_0' \pm t y_0' = 0$ and $x_0'' \pm t y_0'' = 0$, which yields $x_0'' y_0' - y_0'' x_0' = 0$, i.e. $r = 0$ in Eq. (111).

An expression for the stationary-phase contribution $K_{\pm}$ in Eq. (107) valid in this special case can be obtained from Eq. (52). Specifically, expression (111) for the stationary-phase contribution becomes

$$K_{\pm} = \pm F^{-1/3}\, 3^{-1/2}\Gamma(1/3)\,(6\varrho)^{1/3}\, A_{\pm} \exp[\mp i(yt_x - xt_y)/t_y{}^2F^2] \,, \tag{125}$$

where ϱ is defined as

$$\varrho = [(x')^2 + (y')^2]^{5/2}/[y'(y'x''' - x'y''')] \,, \tag{126}$$

and the amplitude function $A_{\pm}$ is given by

$$A_{\pm} = A_1 + O(F^{2/3}) \,, \tag{127}$$

with A_1 given by Eq. (113).

CONTRIBUTION OF THE NONLINEAR TERMS IN THE FREE-SURFACE BOUNDARY CONDITION

The far-field wave-amplitude function K(t) considered in the foregoing corresponds to the usual Neumann-Kelvin theory, in which the free-surface boundary condition is linearized. The far-field wave-amplitude function, K′(t) say, associated with the generalized Neumann-Kelvin theory in which the nonlinear terms in the free-surface boundary condition are taken into account is defined in Noblesse [1] by an expression of the form

$$K'(t) = K(t) + k(t) \,, \tag{128}$$

where k(t) corresponds to the nonlinear terms in the free-surface boundary condition. More precisely, Eqs. (36) and (2) in [1] show that, for a ship with port and starboard symmetry, the function k(t) is given by

$$k(t) = k_+(t) + k_-(t) \quad \text{with} \tag{129}$$

$$k_{\pm}(t) = \iint_f \exp[-i(x \pm ty)/qF^2]\, \chi(\phi)\, dxdy \,, \tag{130}$$

where f represents the positive half of the mean free surface and $\chi(\phi)$ represents the nonlinear terms in the free-surface boundary condition given by

$$\chi(\phi) = [\partial\phi/\partial x - (\nabla\phi)^2/2]\; \partial(\partial\phi/\partial z + F^2\partial^2\phi/\partial x^2)/\partial z \\ - \partial(\nabla\phi)^2/\partial x + \nabla\phi\cdot\nabla(\nabla\phi)^2/2 + O(F^2\phi^3). \tag{131}$$

Let b represent the beam/length ratio of the mean wetted hull and let us assume for simplicity that a line parallel to the x axis in the mean free-surface plane $z = 0$, defined by the equation $y = \eta$ with $0 \leqslant \eta < b/2$, intersects the mean waterline at only two points, defined by $x = \xi_S$ in the stern region and $x = \xi_B$ in the bow region, say (the case when a line $y = \eta$ intersects the mean waterline at more than two points yields a result identical to that obtained below and may be treated similarly). Equation (130) then yields

$$k_\pm(t) = \int_0^{b/2} dy \exp(\mp ity/qF^2) \left[\int_{-\infty}^{\xi_S} \exp(-ix/qF^2)\, \chi(\phi)\, dx + \int_{\xi_B}^{+\infty} \exp(-ix/qF^2)\, \chi(\phi) dx \right] + \int_{b/2}^{+\infty} dy \exp(\mp ity/qF^2) \int_{-\infty}^{\infty} \exp(-ix/qF^2)\, \chi(\phi)\, dx \, .$$

The inner integrals (with respect to x) in the foregoing expression are Fourier integrals for which asymptotic expansions, valid in the limit $qF^2 \to 0$, can easily be obtained, e.g., from Eqs. (19) and (20) or from Eqs. (2) and (3) p. 47 in Erdélyi [4], provided that the free-surface nonlinear term $\chi(\phi)$ and its x derivatives vanish as $|x| \to \infty$, which is presumed. We may then obtain

$$k_\pm(t) \sim iF^2 q \left[\int_0^{b/2} \exp[-i(\xi_S \pm ty)/qF^2]\, [\chi - iF^2 q \chi_x + O(F^4)]_{\xi_S}\, dy - \int_0^{b/2} \exp[-i(\xi_B \pm ty)/qF^2]\, [\chi - iF^2 q \chi_x + O(F^4)]_{\xi_B}\, dy \right],$$

which may be expressed in the form of the line integral

$$k_\pm(t) \sim iF^2 q \int_{\text{bow}}^{\text{stern}} \exp[-i(x \pm ty)/qF^2]\, [\chi - iF^2 q \chi_x + O(F^4)]\, t_y dl \, , \tag{132}$$

where the relation $dy = -t_y dl$ was used.

By using the parametric representation (1a,b), which yields $t_y dl = -y_0' d\lambda$, we may express the waterline integral (132) in the form

$$k_\pm(t) \sim -iF^2 q \int_{\lambda_B}^{\lambda_S} \exp[-i(x_0 \pm t y_0)/qF^2]\, [\chi - iF^2 q \chi_x + O(F^4)]\, y_0' d\lambda \, . \tag{133}$$

Equations (128) and (4)-(8) then show that the far-field wave-amplitude function $K'(t)$ in the generalized Neumann-Kelvin theory may be expressed in the form

$$K'(t) \sim q^2 (K'_+ + K'_-) \quad \text{as} \quad F \to 0 \quad \text{with} \tag{134}$$

$$K'_\pm(t) \sim \int_{\lambda_B}^{\lambda_S} \exp[-i(x_0 \pm t y_0)/qF^2]\, a'_\pm \, d\lambda \, , \tag{135}$$

where the generalized amplitude function $a'_\pm$ is given by

$$a'_\pm = u_\pm a_1^\pm - iF^2 q\, [iq(u_\pm)^2 a_2^\pm + p^2 y_0' \chi] + O(F^4) \, . \tag{136}$$

The low-Froude-number asymptotic expansion (134)-(136) shows that the contribution of the nonlinear term χ in the free-surface boundary condition in the generalized amplitude function $a'_\pm$ is $O(F^2)$, which is asymptotically negligible in comparison with the O(1) term $u_{\pm}a_1^{\mp}$ given by the linear Neumann-Kelvin theory except if $u_{\pm}a_1^{\mp} = 0$. More precisely, Eq. (136) shows that the low-Froude-number asymptotic expansion obtained in the foregoing for the wave-amplitude function K(t) immediately yields a corresponding expansion for the generalized wave-amplitude function K′(t). Specifically, the asymptotic expansion for K(t) given by Eqs. (107)-(119) is valid if K(t) on the left side of Eq. (107) is replaced by K′(t) and if the term $iq(u_{\pm})^2a_2^{\mp}$ in expressions (56) and (86) for the second-order amplitude functions $A_2^{\mp}$ is replaced by the term $iq(u_{\pm})^2a_2^{\pm} + p^2y_0'\chi(\phi)$.

This generalized asymptotic expansion for the function K′(t) shows that we have $k(t) = O(F^3)$, as may be seen from Eqs. (107), (111) and (112). This contribution of the nonlinear terms in the free-surface boundary condition must be compared with the function K(t) corresponding to the linear Neumann-Kelvin theory. Specifically, we have $K(t) = O(F)$ for a ship form having a region of flare, $K(t) = O(F^2)$ for a ship form that is wall sided everywhere and has either a bow or a stern (or both) that is neither cusped nor round, and $K(t) = O(F^3)$ for a wall-sided ship form with both bow and stern that are either cusped or round. The contribution of the nonlinear terms in the free-surface boundary condition is then asymptotically negligible for ship forms having a region of flare or a sharp wedge-like bow or/and stern, which is the case for a majority of ships, notably naval ships.

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